Engineering Notes

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Relativistic Rocket Motion via Minkowski's Formalism

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SINCE the work of Esnault-Pelterie[1] in the 1930s, the dynamics of high-speed rockets, i.e., those at relativistic velocities, has been widely investigated as an application of the special theory of relativity to the motion of particles with changing rest masses. Contributions to this problem have been made especially by Ackeret,[2] Bade,[3] and Krause,[4] and more recently Pomeranz,[5,6] among others.

All of these papers are based on three-dimensional treatment of the problem, allowing for the Lorentz transformation of the quantities in accordance with the two postulates of Einstein's relativity theory and, as a result, leading to nonlinear relations for the velocity of the particle. According to Ref. 4, direct integration of the equation of motion has been shown to be feasible in a few cases, always with the assumption of colinear velocity and acceleration.

The purpose of this Note is to present a full four-dimensional formulation of the problem following the line of Arzelies[7,8] and to show that, by means of a slightly different approach, the four-dimensional equation of motion can be put into a quasilinear form. Previous results are thus easily derived from the related general solution, namely Ackeret's equation.

Four-Dimensional Equation of Motion

From a technical point of view, a rocket or a spacecraft undergoing propulsive effects can be seen as a particle with changing rest (or proper) mass as far as the motion of its center of inertia is concerned.

If an isolated particle (i.e., one without external forces acting on it) having instantaneous proper mass m_0 and four velocity V in an observer's frame undergoes some exchange of proper mass with the surrounding medium, this exchange may be:

1) Homokinetic, if the variation of proper mass does not affect the state of motion of the particle, as stated in Ref. 7.

2) Heterokinetic, if the variation of proper mass modifies the state of motion of the particle, as stated in Ref. 8.

3) Mixed, when the variation of proper mass takes place according to both of the preceding modes.

If a mixed exchange of proper mass is assumed and the homokinetic and the heterokinetic contributions are $\mathrm{d}m_{0V}$ and $\mathrm{d}m_{0W}$, respectively, between the instants τ and $(\tau+\mathrm{d}\tau)$ measured in the rest frame of the particle, the theorem of conservation of four momentums reads,

$$(m_0+\mathrm{d}m_0)(V+\mathrm{d}V)-\mathrm{d}m_{0V}V-\mathrm{d}m_{0W}W=m_0V \tag{1}$$

where $\mathrm{d}m_{0V}$ and $\mathrm{d}m_{0W}$ are assumed positive masses if they enter into the particle, and W represents the four velocity of the heterokinetic mass $\mathrm{d}m_{0W}$ in the observer's frame.

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Equation (1) is written in Minkowski's formalism as,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(m_0V)=\frac{\mathrm{d}m_{0V}}{\mathrm{d}\tau}V+\frac{\mathrm{d}m_{0W}}{\mathrm{d}\tau}W \tag{2}$$

If Eq. (2) is considered in the particle rest Lorentz frame K^*, the timelike part of the equation becomes

$$\mathrm{d}m_0=\mathrm{d}m_{0V}+\frac{\mathrm{d}m_{0W}}{\alpha_W^*} \tag{3}$$

where $\alpha_W^*=[1-(W^*/c)^2]^{1/2}$.

Thus, in the observer's Lorentz frame K, letting $\mathrm{d}m_{0W}^*/\mathrm{d}\tau=\mathrm{d}m_{0W}/(\alpha_W^*\mathrm{d}\tau)$,

$$m_0\frac{\mathrm{d}V}{\mathrm{d}\tau}=\frac{\mathrm{d}m_{0W}^*}{\mathrm{d}\tau}(\alpha_W^*W-V) \tag{4}$$

Equation (4) can also be written

$$m_0\dot{V}=(\dot{m}_0-\dot{m}_{0V})(\alpha_W^*W-V) \tag{5}$$

where the dot stands for the particle proper time derivative.

Both Eqs. (4) and (5) are vector differential equations in Minkowski space and are not linear with regard to V because of the α_W^*W term. Let us then investigate this term in order to see whether some completely linear differential equation can finally be written.

One can consider all four vectors as belonging to an Euclidean Minkowski's manifold over the body of complex numbers. If, without lack of generality, a special Lorentz transformation holds for the components of W, these are, taking into account that $V=(c/\alpha,\ iv/\alpha)$, with $\alpha=[1-(v/c)^2]^{1/2}$,

$$\begin{bmatrix} W_0\\ W_1\\ W_2\\ W_3\end{bmatrix}=(1/c)\begin{bmatrix} W_0^* & -W_1^* & 0 & 0\\ W_1^* & W_0^* & 0 & 0\\ 0 & 0 & W_0^* & 0\\ 0 & 0 & 0 & W_0^*\end{bmatrix}\begin{bmatrix} V_0\\ V_1\\ 0\\ 0\end{bmatrix}+\begin{bmatrix} 0\\ 0\\ W_2^*\\ W_3^*\end{bmatrix} \tag{6}$$

that is, in condensed form,

$$\{W\}=(W_0^*/c)[I]\{V\}+[T]\{V\}+\{W_\perp^*\} \tag{7}$$

where $w_\perp^*\cdot V\equiv 0$ and

$$[T]=(1/c)\left[\begin{array}{cc|c} 0 & -W_1^* & 0\\ W_1^* & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0\end{array}\right] \tag{8}$$

It can be shown that for a general Lorentz transformation involving parallel axis frames, matrix $[T]$ and vector $W^*_\perp$ generalizes to

$$[T]=[1/(\alpha^*_W c)]\left[\begin{array}{c|c} 0 & -iw^* \\ \hline iw^*_\parallel & 0 \end{array}\right],$$

$$\{W^*_\perp\}=(1/\alpha^*_W)\left[\begin{array}{c} 0 \\ \hline iw^*_\perp \end{array}\right]$$

These last expressions, then, represent matrix operators in Minkowski space which behave like tensors for general spacelike rotations. Therefore, Eq. (5) can now be written in canonical form as,

$$\dot{V}=[(\dot{m}_0-\dot{m}_{0V})/m_0](\alpha^*_W TV+W^*_\perp) \tag{9}$$

Equation (9) looks deceptively simple; it readily shows a "linear" structure with regard to V, with variable coefficients either in the quantity between the brackets and/or the operator T and vector $W^*_\perp$.

In fact, the general case happens to be nonlinear because of the quantities $w^*_\parallel$ and $w^*_\perp$, despite the possibility of w^* being a known vector function of τ. Only approximate solutions and iteration procedures seem appropriate to predict the rocket motion using Eq. (9) in such a general case. One line of approach could be a quasidynamic solution built stepwise around a "stationary" solution by assuming fixed values for $w^*_\parallel$ and $w^*_\perp$, their values being updated at the start of each following step. According to Refs. 9 and 10, the stepwise general solution is,

$$V=e^{B(\tau)}\left[e^{-B(0)}V(0)+\int_0^\tau e^{-B(s)}u(s)\,\mathrm{d}s\right] \tag{10}$$

where

$$B(\tau)=\int_0^\tau[(\dot{m}_0-\dot{m}_{0V})/m_0]\,T(s)\alpha^*_W(s)\,\mathrm{d}s$$

$$u(s)=[(\dot{m}_0-\dot{m}_{0V})/m_0]\,W^*_\perp(s)\alpha^*_W(s)$$

and

$$e^B=\sum_{n=0}^{\infty}(B^n/n!)$$

This general solution is allowed by the commutativity of matrices B and T, which can be demonstrated by means of the hermitian nature of the projection operator of w^* over v^*.

Ackeret's Equation and Solution

Starting with Eq. (4) and restricting the analysis to the case of a rectilinear trajectory, the foregoing equations can be written, obviating some calculation details as,

$$m_0\mathrm{d}u=\alpha^2\mathrm{d}m^*_{0W}\left(\frac{\alpha\alpha^*_W}{\alpha_W}s-u\right) \tag{11}$$

where u and s are the rocket and the ejected mass three-dimensional velocities in observer's frame, respectively, and $\alpha_W=\sqrt{1-(s/c)^2}$, $\alpha=\sqrt{1-(u/c)^2}$.

Bearing in mind the relation $(1/\alpha^*_W)=[1/(\alpha_W\alpha)(1-\beta\beta_w)]$ and the relativistic law of velocities transformation, Eq. (11) can be further transformed as,

$$m_0\mathrm{d}u=\alpha^2 s_0\mathrm{d}m^*_{0W} \tag{12}$$

Equation (12) represents an equivalent form of Ackeret's equation as found in Ref. 11 with slight notation differences.

As can be seen, Eq. (12) is independent of the kind of mass exchange undergone by the rest mass of the rocket, and only the heterokinetic part is taken into account. However, in a mixed case, the heterokinetic part must be calculated according to Eq. (3), showing that a better physical understanding of the whole process is achieved from Eq. (5). The inference just developed was to check the internal consistency of the present formulation. Thus, direct management of Eq. (9) must lead to Ackeret's solution in the rectilinear case.

In effect, let us assume that a heterokinetic case and Eq. (9) reads

$$\dot{V}=\frac{\mathrm{d}}{\mathrm{d}\tau}(\ln m_0)\alpha^*_W TV \tag{13}$$

Equation (13) represents an homogeneous linear differential equation with variable coefficients (allowing w^* to be a known function of τ).

The general solution of Eq. (13) is, according to Ref. 10,

$$V=\exp\{(\ln m_0)T(\tau)\alpha^*_W(\tau)-(\ln M_0)T(0)\alpha^*_W(0)\}V(0) \tag{14}$$

where M_0 is the rest mass of the rocket at initial proper time $\tau=0$.

Assume in addition, as all other authors did, a constant velocity for the ejected masses in the rocket rest frame; writing $v=0$ for the initial proper time, the general solution reads in matrix form

$$\begin{bmatrix} V_0 \\ V_1 \end{bmatrix}=\exp\left\{(1/c)\ln(m_0/M_0)\begin{bmatrix} 0 & -iw^* \\ iw^* & 0 \end{bmatrix}\right\}\begin{bmatrix} c \\ 0 \end{bmatrix} \tag{15}$$

By performing the power expansion of the matrix exponential function, and rearranging terms, one has for V_1

$$V_1=ic\sinh\{(w^*/c)\ln(m_0/M_0)\}$$

The three-dimensional velocity can finally be written, solving first for β

$$\frac{v}{c}=\frac{1-(1/r)^{2w^*/c}}{1+(1/r)^{2w^*/c}}$$

This is Ackeret's solution as presented in Ref. 4. It was inferred in this paper by direct integration of the four-dimensional equation of motion for the particular case when $v\|w$.

Conclusions

A phenomenological analysis of the motion of particles with variable rest masses leads to a general equation of motion for such a physical system in Minkowski's formalism. The use of general Lorentz transformation allows for a writing of that equation in a quasilinear form; accordingly, approximate stepwise general solutions can be obtained in order to build an integration procedure for the general case, although further investigations are still needed.

A completely linear four-vector differential equation is obtained when the rocket and exhaust gas velocities are colinear; in this case, as expected, reworking of the equations leads to Ackeret's equation of motion. Nevertheless, the related solution can be obtained by direct integration of the linear differential system for four-vector quantities. Ackeret's solution becomes just the tridimensional part of that Minkowski space-based solution.

Thinking and working in Minkowski's formalism seem to be less cumbersome for relativistic dynamics analysis than former formulations. The mathematical expressions show a complete transparency with regard to the physics of the involved processes when such processes are described in a consistent four-dimensional way. Finally, the four-dimensional equations are automatically written in covariant form, which promises future progress by studying the subject in the proper accelerated frame of the rocket.

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Nonstationary Shaping Filters for Simulation of Gravity Uncertainty Effects on Missile Trajectories

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Introduction

IT has been recognized for some years that imperfect knowledge of the detailed structure of the Earth's gravity is a major source of error in very precise inertial navigation and guidance systems.[1] To analyze the accuracy of such systems and to process test data optimally, suitable statistical models are needed to account for the effects of gravity uncertainty. The suitability of models is governed by two principal factors. For physical realism, the models need to be "self consistent,"[2] i.e., the joint statistics of the gravity disturbance vectors at different points in space must satisfy the mathematical constraints of potential field theory. For convenience of use in system simulation and Kalman filtering/smoothing software, the models need to be realized in state-space form.

A variety of gravity statistics models were recently reviewed by Heller.[3] This discussion reveals that the factors just named have not proved to be entirely compatible. The mathematics of self-consistency require auto- or cross-correlation functions which are not realizable exactly in state-space form. Consequently, the practice has been to formulate a fully self-consistent model in analytic form and subsequently to approximate the required auto- and cross-correlations by a state-space shaping filter.

The methods used to form state-space approximations have become progressively more sophisticated and powerful.[4,5] However, to date, all of them require that the model statistics be stationary. This is an acceptable limitation for modeling gravity statistics at a fixed altitude, but is seriously violated when the altitude must change significantly, as, for example, along a missile trajectory. As altitude changes, the gravity statistics must be "upward continued,"[6] producing significant changes in the variance, correlation length, and correlation function shape. Previous state-space gravity modeling methods were unable to deal with these changes.

This Note presents a state-space modeling concept which can be used to approximate the desired type of nonstationarity. This is accomplished by initializing the state-space model so that its second-order statistics exhibit a transient response. The model parameters and initial condition are adjusted to fit the transient response to the statistics prescribed by a nonstate-space analytic statistical model for gravity. In this Note, the general form of the model equations is first derived and several low-order examples are given explicitly. A first-order model is subsequently fitted to the attenuated white noise gravity statistics model[7] to demonstrate the application of the concept.

Nonstationary State-Space Models

General Formulas

A state-sapce model, or shaping filter, is a stochastic linear dynamic system of the form

$$\dot{x}(t)=Fx(t)+w(t) \qquad (t\geq 0) \tag{1}$$

where

$$E\{x(0)x^T(0)\}\triangleq P_0 \tag{2}$$

$$E\{w(t_1)w(t_2)\}\triangleq Q\delta(t_2-t_1) \tag{3}$$

($E\{\cdot\}$ denotes statistical expectation, and $\delta(\cdot)$ the delta function.)

For the present application, the state dynamics matrix F is required to have all eigenvalues in the left half of the complex plane. For this model, it can be shown (see Appendix) that the correlation between the states at times t_1 and t_2 (both ≥ 0) is given by

$$C(t_1,t_2)\triangleq E\{x(t_1)x^T(t_2)\}=\Phi(t_1)[P_0-P_\infty]\Phi^T(t_2)+\Phi(t_1-d)P_\infty\Phi^T(t_2-d) \tag{4}$$

where $\Phi(t)=e^{Ft}$ = state transition matrix and P_∞ = steady-state covariance matrix solving,

$$FP_\infty+P_\infty F+Q=0 \tag{5}$$

$$d=\min[t_1,t_2] \tag{6}$$

A straightforward consequence of Eq. (4) is that the state covariance matrix at time t is given by

$$P(t)\triangleq C(t,t)=\Phi(t)[P_0-P_\infty]\Phi^T(t)+P_\infty \tag{7}$$

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